Sub-pupil estimation of the laser guide star tilt term

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ABSTRACT

We present a new technique aimed to obtain the downward tilt of a Laser Guide Star (LGS). The concept of this technique consists in averaging the LGS tilt signals as seen from many non-contiguous sub-pupils located on the telescope aperture. The sub-pupils tilt signals have two different contributions. One, common to all the various sub-pupils, is due to the upward beam propagation, the other, different on the various sub-pupils, is due to the downward beam propagation. These downward tilts have a correlation degree dependent on the sub-pupils array geometry and on the value of outer scale of turbulence. Summing up the sub-pupil signals allows to average out the tilt signal component due to the beacon downward propagation, so obtaining an estimation of the upward propagation laser beam tilt.

We show here the achievable results in terms of the ratio between residual tip-tilt error variance obtained using this technique and the uncorrected tip-tilt variance. In particular the dependence from outer scale of turbulence and the use of multi-telescope systems are investigated.

Keywords: adaptive optics, laser guide star, tilt estimation

1. INTRODUCTION

The lack of absolute tip-tilt knowledge in the Laser Guide Star is a well known problem attacked by several authors in a number of different ways. A few schemes, however, are not able to solve in a definitive way such a problem, but can limit its extent in term of reduction of image degradation due to the uncorrected jitter and/or in terms of expansion of the useful sky coverage. For instance the Double Adaptive Optics extends significantly the sky coverage (but being still far from the whole sky-coverage in the visible wavelengths) at the expenses of the duplication of the adaptive optics system.

In this paper we outlines a simple technique able to reduce of a modest factor the residual tip-tilt jitter. The basic idea is to isolate some portions of the input pupil chosen as to have a higher degree of uncorrelation, higher than the one deriving from the integration over the whole entrance aperture of the telescope. The average of their tilt signals is a better estimation of the up-ward tilt of a Laser Guide Star because the down-ward term is hopefully cancelled out by the averaging operation.

Extending the technique to the multiple aperture large telescopes like the Large Binocular Telescope (LBT), the Keck I&II and the Very Large Telescope Interferometer (VLTI) somewhat better performances can be achieved provided that atmospheric parameters, especially the outer scale $L_0$, shows favorable values.

We wish to point out that in the following treatment the performance are compared with the ones that one could obtains without any attempt to correct the tip-tilt signal. As it has been already pointed out by Winker and Bonaccini, very low $L_0$ values leads to very tiny tip-tilt terms such that, in most cases, no correction is required.
2. THE SUB-PUPIL AVERAGING TECHNIQUE

The LGS tilt (x or y component) measured by the main telescope is given by

\[ \theta_0^{(m)} = \theta_0 - \theta_{up}, \]  

where \( \theta_0 \) and \( \theta_{up} \) are respectively the tilt associated to the downward and upward propagation of the beam. If we measure the tilt \( \theta_i^{(m)} \) by some auxiliary aperture, we can obtain similar relations for \( \theta_i^{(m)} \)

\[ \theta_i^{(m)} = \theta_i - \theta_{up}. \]

The average of the tilts measured by \( N \) auxiliary apertures is

\[ \frac{\sum_{i=1}^{N} \theta_i^{(m)}}{N} = \frac{\sum_{i=1}^{N} \theta_i}{N} - \theta_{up}, \]

and can be used with the eq. (1) to obtain the following expression for \( \theta_0 \)

\[ \theta_0 = \frac{\sum_{i=1}^{N} \theta_i}{N} + \theta_0^{(m)} - \frac{\sum_{i=1}^{N} \theta_i^{(m)}}{N}. \]  

The quantity \( \sum_{i=1}^{N} \theta_i/N \) is unknown, but, if the tilts \( \theta_i \) are decorrelated, we can suppose to extend the sum over an enough large number \( N \) of auxiliary apertures in order to obtain a small contribution of the previous quantity respect to \( \theta_0 \). If \( \sum_{i=1}^{N} \theta_i/N \) is neglected in eq. (2), we can estimate \( \theta_0 \) obtaining an error residual variance \( \sigma_{\theta_0}^2 \), given by

\[ \sigma_{\theta_0}^2 = \frac{\left( \sum_{i=1}^{N} \theta_i \right)^2}{N} = \frac{\left( \theta_0^2 \right)}{1/N + 2/N^2 \sum_{i=1}^{N-1} \sum_{j>i+1} \langle \theta_i \theta_j \rangle - \langle \theta_i \rangle^2}, \]  

where \( N \) auxiliary apertures with the same diameter \( D \) are considered (so \( \langle \theta_1^2 \rangle = \langle \theta_2^2 \rangle = \ldots = \langle \theta_N^2 \rangle \)). A trivial analysis of eq. (3) suggests to use a large number \( N \) of auxiliary telescopes around the main telescope in order to reduce the variance term, and also to separate them enough in order to reduce the covariance contribution. This solution is hardware expensive and have to solve the problem of differential tracking among the auxiliary telescopes, that induces an error on the estimation of \( \theta_0 \). By the way, it could be applicable in the case of telescope arrays that are going to implement in the near future, such as VLTI, Keck I&II or LBT. The LBT has the advantage to have two mirrors on the same mounting, so that the problem of differential tracking is cancelled out, even if (as in the case of Keck) the low number of apertures \( N = 2 \) doesn’t permit to reduce the error \( \sigma_{\theta_0}^2 \) much below \( 0.5 \langle \theta_1^2 \rangle \).

Another approach is given by the use of sub-pupils of the main telescope as auxiliary apertures. This is a zero-cost solution, because it is sufficient to use the signals given by the border lenslets of the Shack-Hartman sensor (if it is used as high-order wavefront sensor), or to put a mask on an image of the main telescope pupil in the beam path that reach the tip-tilt sensor (provided the bandwidth is large enough to allow non-coherent superposition). In order to maximize the number \( N \) of sub-apertures, we cannot set them covering the whole telescope aperture. In this case, the average tilt \( \sum_{i=1}^{N} \theta_i/N \) approaches the tilt \( \theta_0 \) of the main telescope, so \( \sigma_{\theta_0} \approx \theta_0 \), making the technique ineffective.

In this paper we deal in the configuration shown in fig. 1, in which the sub-pupils are within a ring of width \( D \) at the border of the main telescope aperture. If the number \( N \) of sub-pupils increases when their diameter \( D \) is fixed, the error contribution of the variances goes down, but the minimum distance among the pupils decreases and the contribution of the correlations grows up. We can deduce that an optimal configuration exists for \( D \) and \( N \), giving the minimum value for the ratio \( \gamma \) between the residual tip-tilt error variance \( \sigma_{\theta_0,x}^2 + \sigma_{\theta_0,y}^2 \) and the tip-tilt variance \( 2 \langle \theta_0^2 \rangle \) without any correction. With the use of eq. (3) we can write

\[ \gamma = \frac{\sigma_{\theta_0,x}^2 + \sigma_{\theta_0,y}^2}{2 \langle \theta_0^2 \rangle} = \frac{\langle \theta_0^2 \rangle}{\langle \theta_0^2 \rangle} \left( \frac{1}{N} + 2/N^2 \sum_{i=1}^{N-1} \sum_{j>i+1} \langle \theta_i \theta_j \rangle + \langle \theta_i \rangle^2 \right). \]  

In the next section we’ll find the expression for variances and covariances needed to compute the parameter \( \gamma \).
Figure 1. The $N$ sub-pupils have diameter $D$. They are within a ring at the border of the main aperture of diameter $D_0$.

Figure 2. The LGS is generated at the altitude of $H = 90\text{km}$. The position of the center of $i$-th sub-pupil is defined by the vector $\mathbf{r}_i$ on the plane of the main aperture. The radiation from the LGS propagates to the sub-pupil sampling the turbulence layer at height $z$ in the circular portion with center defined by $\mathbf{r}_i'$. The same for the $j$-th sub-pupil.
3. CONE-TO-CONE CORRELATIONS

Fig. 2 shows the geometry of the system. We consider the case of a LGS generated by a laser tuned to the sodium D-line and focused on the mesospheric layer at the altitude of $H = 90$ km. The position of the $i$-th sub-pupil is identified by a vector $\mathbf{r}_i = (x_i, y_i)$ on the plane of the main telescope aperture of diameter $D_0$. The light from the LGS reaches the sub-apertures sampling the turbulent atmosphere with cones having the same origin on the LGS itself. The circular portion the turbulent layer at height $z$ intersected by the cone associated with the $i$-th sub-pupil induces a phase fluctuation that can be expanded in terms of Zernike polynomials.\(^{11}\) We let $a_{x,i}(z)$ and $a_{y,i}(z)$ respectively the $x$ and $y$ Zernike tilt coefficients.

Takato and Yamaguchi\(^{12}\) give the expression for the covariance between the Zernike coefficients of two displaced circular apertures in the case of finite outer scale. With the use of our notation, the tip-tilt covariance between the phase disturbances in the $i$-th and $j$-th portions of the turbulent layer at height $z$ can be written as

$$
(a_{x,i}, a_{x,j}) + (a_{y,i}, a_{y,j}) = 2^5 \pi^{8/3} 0.00969 \left( \frac{2\pi}{\lambda} \right)^2 C_n^2(z) \Delta z D' \frac{I_{022}(2 r_{ij}' / D', \pi D'/L_0)}{L_0} ,
$$

where

$$
I_{\kappa\mu\nu}(\alpha, \alpha_0) = \int_0^\infty \frac{J_\kappa(\alpha s)J_\mu(s)J_\nu(s)}{s(s^2 + \alpha_0^2)}^{11/6} ds,
$$

and $\lambda$ is the wavelength, $C_n^2(z)$ is the refractive index structure constant profile, $\Delta z$ is the thickness of the layer, $L_0$ is the outer-scale of the turbulence and $J_\beta$ is the Bessel function of order $\beta$. The quantities $D'$ and $r_{ij}'$ are respectively the diameter of the circular portion of the layer sampled by a sub-pupil and the displacement between the $i$-th and the $j$-th portion at the height of the layer. The quantities $D'$ and $r_{ij}'$ are related to $D$ and $r_{ij} = |r_i - r_j|$ by the simple scaling law

$$
D' = D \left(1 - \frac{z}{H}\right) \quad r_{ij}' = r_{ij} \left(1 - \frac{z}{H}\right)
$$

Substituting the previous relations in the eq. (5) and integrating the contribution of all turbulent layers, we obtain

$$
(a_{x,i}, a_{x,j}) + (a_{y,i}, a_{y,j}) = 2^5 \pi^{8/3} 0.0229 \left( \frac{D}{r_0} \right)^{5/3} \int_0^H c_n^2(z) \left(1 - \frac{z}{H}\right) I_{022} \left[2 r_{ij}' / D', \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz
$$

where $c_n(z)$ is the normalized profile defined as

$$
c_n^2(z) = \frac{C_n^2(z)}{\int_0^H C_n^2(z) dz}
$$

and $r_0$ is the Fried parameter for a plane wave defined by

$$
r_0 = \left[0.423 \left(\frac{2\pi}{\lambda}\right)^2 \int_0^\infty C_n^2(z) dz\right]^{-3/5}
$$

Eq. (8) reduces to tip-tilt variance when $r_{ij} = 0$.

The relation between the angular tilt $\theta$ ($x$ or $y$ component) and the corresponding Zernike phase coefficient is given by

$$
\theta = \frac{2 \lambda}{\pi D} a
$$

With the use of eqs. (8) and (11) we can write the following expression for $\gamma$

$$
\gamma = \left(\frac{D_0}{D}\right)^{1/3} \frac{J_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[0, \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz}{\int_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[0, \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz}
$$

$$
\times \left\{ \frac{1}{N} + \frac{2}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{J_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[2 r_{ij}' / D', \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz}{\int_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[0, \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz} \right\}
$$

$$
= \frac{1}{N} + \frac{2}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{J_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[2 r_{ij}' / D', \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz}{\int_0^H c_n^2(z) \left(1 - \frac{z}{H}\right)^{5/3} I_{022} \left[0, \pi \left(1 - \frac{z}{H}\right) D'/L_0\right] dz}
$$

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4. NUMERICAL RESULTS

4.1. Single-aperture 8m-class telescope example

Once the $C_n^2(z)$ profile has been defined, we can numerically evaluate, using eq. (12), the ratio $\gamma$ between the residual tip-tilt error variance and the uncorrect tip-tilt variance. In fig. 3 the profile we use is shown. It's a Hufnagel-Valley profile scaled to give $r_0 = 20$ cm at $\lambda = 0.55 \mu m$.

The outer scale $L_0$ is a wide spread parameter in literature ranging from a few meters to about 100 m. Because the statistical properties of the refractive index fluctuations at spatial scale greater than the outer scale depends on the nature of the energy source of the turbulence (in particular homogeneity and isotropy are not further ensured) the given formula are not suitable for the range of outer scale $L_0 < D_0$. In any case we give the results obtained for that range in order to compare them with the results by other authors.

In fig. 4 the behavior of the tip-tilt Strehl ratio attenuation and the angular resolution obtained without any correction are shown as depending on outer scale $L_0$, for a telescope of diameter $D_0 = 8.4$ m. Very low $L_0$ values leads to reduce the tilt components so that tilt correction becomes less crucial respect to high-order correction.

In fig. 5 the dependence of the $\gamma$ ratio on the number $N$ and the diameter $D$ of the sub-pupils is shown. The case with $D_0 = 8.4$ m and $L_0 = 50$ m is considered. A simple scaling of the results is possible because $\gamma$ depends on $D$, $D_0$ and $L_0$ only by means of the $D/L_0$ and $D_0/L_0$ ratios (see eq. (12)). In the figure the point of best correction is emphasized by a triangular mark, that gives $\gamma = 0.73$ for $D_{opt} = 30$ m and $N_{opt} = 84$. The $\gamma$ value is less then one for a wide range of $D$ and $N$ values, even if, as it has been pointed out in the previous sections, the $\gamma$ ratio grows up when $N < N_{opt}$ because of the contribution of variance terms, and when $N > N_{opt}$ because of the covariance terms.

The best values of $\gamma$ has been calculated with different values of the outer scale and the results are shown in fig. 6. An optimal application of the sub-pupil technique can be obtained when $L_{0,opt} \approx 40$ m ($\gamma = 0.72$), but the $\gamma$ ratio is weakly dependent on the outer scale if $L_0 \geq L_{0,opt}$. The technique is ineffective for low values of the outer scale ($\gamma > 1$) because the uncorrected tip-tilt variance tends rapidly towards zero in this range.

4.2. Multi-aperture 8m-class telescope example

The sub-pupil technique can be generalized for telescope arrays, such as VLTI, Keck I&II or LBT. Each telescope of the array use $N$ sub-pupils with the same geometry described in the previous section. Letting $n$ the number of
Figure 4. Solid line: long-exposure angular tilt variance as a function of the outer scale $L_0$. Dashed line: tip-tilt Strehl ratio attenuation.

Figure 5. $\gamma$ ratio between corrected and uncorrected tip-tilt variance as a function of the number $N$ of sub-pupils. Each curve is computed with a constant value of the diameter $D$ of the sub-pupils. Only the curves with $D = 20, 30, 39, 49, 59, 68, 78$ cm are shown. The minimum (best) value of $\gamma$ is emphasized by a triangular mark, giving $\gamma = 0.73$. 
Figure 6. Best values of $\gamma$ ratio between corrected and uncorrected tip-tilt variance as a function of the outer scale $L_0$. The most favorable condition is obtained with $L_0 = 40$ m, giving $\gamma = 0.72$.

main apertures of the telescope array, we can average $nN$ tilt signals, obtaining a reduction of the variance terms of a factor $1/n$, and only a small contribution to the variance is added because the correlation among sub-pupils of different telescopes are expected to be negligible. Letting $\theta_i^{(k)}$ the downward tilt associated with the $i$-th sub-pupil of the $k$-th telescope, we obtain the following expression for the $\gamma$ ratio

$$
\gamma = \frac{\langle \theta_i^2 \rangle}{\langle \theta_i^2 \rangle} \left\{ \frac{1}{n} \left[ \frac{1}{N} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\langle \theta_i^{(1)} \theta_j^{(1)} \rangle}{2 \langle \theta_i^{(1)} \rangle} \right] \right\} 
$$

The contribution of the term within the square brackets is the single-telescope residual error described in the previous section. In the present configuration this error contribution is reduced by a factor $1/n$.

As numerical example, we deal in the LBT case, a two 8.4m-aperture telescope with centers displaced by 14.4 m. In fig. 7 the dependence of the $\gamma$ ratio on the number $N$ of sub-pupils per main aperture and the diameter $D$ of the sub-pupils themselves is shown in the case with $L_0 = 50$. The dashed horizontal line represents the $\gamma$ value obtained by the use of the full main apertures as “sub-pupils” ($N = 1$ and $D = D_0$). Again, the minimum value is emphasized by a triangular mark, giving the optimal value $\gamma = 0.36$ when $D_{\text{opt}} = 30$ cm and $N_{\text{opt}} = 84$. The curves plotted in fig. 7 are similar to the curves shown for the single-telescope case (if rescaled by a factor $1/n = 0.5$). That confirms that the correlations among the sub-pupils in different main apertures are negligible.

In fig. 8 the dependence of the best value of $\gamma$ on the outer scale $L_0$ is shown. Again, we can notice the presence of a most favorable outer scale condition when $L_0 = 33$ m, giving $\gamma = 0.35$. The low $L_0$ behavior differs from the single-aperture one because the multi-aperture case allows to average the tilt signals measured by the whole main apertures.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{1-mirror configuration $D_0 = 8.4$ m}
\end{figure}
Figure 7. $\gamma$ ratio between corrected and uncorrected tip-tilt variance as a function of the number $N$ of sub-pupils per main aperture. The LBT configuration is shown. Each curve is computed with a constant value of the diameter $D$ of the sub-pupils. Only the curves with $D = 20, 30, 39, 49, 59, 68, 78$ cm are shown. The minimum (best) value of $\gamma$ is emphasized by a triangular mark, giving $\gamma = 0.36$. The dashed horizontal line represents the $\gamma$ value obtained by the use of the full main apertures as "sub-pupils" ($N = 1$ and $D = D_0$).

Figure 8. Best values of $\gamma$ ratio between corrected and uncorrected tip-tilt variance as a function of the outer scale $L_0$, in the LBT configuration. The most favorable condition is obtained with $L_0 = 33$ m, giving $\gamma = 0.35$. 
5. CONCLUSIONS

The major advantage of the sub-pupil averaging technique, in spite of its limited capability, is the easiness of implementation. The tilt signals can be provided by the border lenslets of a Shack–Hartman sensor if it is used as high-order wavefront sensor. Otherwise a simple mask placed in a pupil reimager placed in the optical train of a tip–tilt sensor is enough to establish the described scheme. There is no need, in fact, to estimate the tip–tilt of the several sub-apertures separately and to co-add later in some computer system, even if the diffractive effects have to be further studied in this configuration. The photon–counting hardware is unchanged with respect to a NGS tip–tilt sensor, or the introduced read–out noise, if a CCD–type sensor is used, retains its original value. In addition it appears very easy to change the masks in a way to match the right $L_0$ figure. Finally it has been pointed out that current or operating in near–term arrays of large telescopes can achieve even better results with an essentially zero additional cost.

REFERENCES